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DEGENERATE PERIODIC MOTIONS IN THE CASE OF THE GENERATING SET OF QUASI-PERIODIC SOLUTIONS*

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The problem of existence and stability in the small of periodic solutions of a system of ordinary differential equations with a small parameter μ, which in the generating approximation admits a stable in the large set of quasi-periodic solutions, is considered. Four groups of diverse character criteria of periodic solution stability, which differ by the synchronization of not all generating rapid phases, are obtained. Dependence of the considered here solutions on remaining phases, called quasi-static, are completely absent.

The particular nondegenerate case of synchronization of all rapid phases was considered previously in /1/, other particular aspects of the general problem considered here were investigated in /2—5/. The technique of derivation of periodic solution in the form of series in powers of the small parameter was developed by Poincaré and others /2/.

1. Structure of the problem generating system. In applications related to celestial mechanics and technology of interest is the problem of periodic solutions of the following almost conservative dynamic system:

$$p^{\cdot} = -\partial H/\partial q + \mu Q,\ q^{\cdot} = \partial H/\partial p,\ u^{\cdot} = U \tag{1.1}$$
$$H = H(q, p),\ Q = Q(q, p, u, \psi, \mu)$$
$$U = U(q, p, u, \psi, \mu)$$

where p and q are, respectively, the row and column vectors of conjugate canonic variables, u is the vector of remaining coordinates of the system, and the generating Hamiltonian H and the vector functions Q and U are periodic with respect to the phase of external perturbations $\psi = \nu t$ and analytic with respect to remaining variables. It is assumed that in the generating approximation $(\mu = 0)$ the isolated conservative subsystem admits a general quasi-periodic solution. The corresponding to it rapid phases will be conditionally separated in two groups

$$\psi_i = \omega_i t + \alpha_i,\ i = 1, \ldots, l; \tag{1.2}$$
$$\varphi_s = \Omega_s t + \beta_s,\ s = 1, \ldots, m$$

with the first and second group phases linked by the action constants $I_1, \ldots, I_l$ and $K_1, \ldots, K_m$. The phase space of the canonical generating subsystem is, thus, completely filled by tori with $l + m + 1$ at angle coordinates

$$p = p(\psi_1, \ldots, \psi_l, I_1, \ldots, I_l, \varphi_1, \ldots, \varphi_m, K_1, \ldots, K_m, \theta_1, \ldots, \theta_n, h_1, \ldots, h_n, \psi) \tag{1.3}$$
$$q = q(\psi_1, \ldots, \psi_l, I_1, \ldots, I_l, \varphi_1, \ldots, \varphi_m, K_1, \ldots, K_m, \theta_1, \ldots, \theta_n, h_1, \ldots, h_n, \psi)$$

where $\theta_1, \ldots, \theta_n, h_1, \ldots, h_n$ are the remaining conjugate constants of integration, so that $l + m + n$ is the dimension of vector p (or q). The first group partial frequencies $\omega_1, \ldots, \omega_l$ are real mutually independent functions of action variables, and

$$\omega_i = \partial H/\partial I_i \tag{1.4}$$

In terms of action variables the generating Hamiltonian is of the form $H = H(I_1, \ldots, I_l, K_1, \ldots, K_m)$, which shows that the matrix coefficient of slope

$$e_{ij} = \partial\omega_i/\partial I_j = \partial^2 H/\partial I_i \partial I_j \tag{1.5}$$

is of rank l. The second group partial frequencies $\Omega_1, \ldots, \Omega_m$ are also mutually independent and satisfy the equalities

$$\Omega_s = \partial H/\partial K_s \tag{1.6}$$

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However among them there may appear isochronous phases (Ω_s = const) which will be assumed highly incommensurable with the perturbation frequency ν.

These frequencies are, moreover, such that the quantities

$$\infty > \Omega_s|_{K_1 = \ldots = K_m = 0} > 0 \tag{1.7}$$

are not small. Note, also, that the frequencies that are not isochronous for $K_1 = \ldots = K_m = 0$ may prove to be equal. The isochronous phases in the general solution (1.3) may not belong to the second group. It is assumed that the generating system is such (or can be made such) that the frequencies corresponding to these phases are identically equal ν, and can consequently be related to the external phase ψ which explicitly appears in (1.3). In such case the generating Hamiltonian in formulas (1.4)—(1.6) is determined with an accuracy to the constant term νI, where I is the action constant conjugate of the isochronous resonance phase. Finally, note that the fulfillment of inequalities of the type of (1.7) is not necessary for frequencies of the first group.

In what follows we call phases of the first group, and their actions and phase shifts $\alpha_1, \ldots, \alpha_l$, anisochronous, and the corresponding characteristics of the second group, quasi-static.

One more important assumption relates to the form of dependence of the general solution (1.3) on quasi-static phases and actions. Let us assume that the variable quantities p and q are analytic functions of the quantities

$$\lambda_s = \sqrt{K_s}\cos\varphi_s, \quad \mu_s = \sqrt{K_s}\sin\varphi_s \tag{1.8}$$

The last of vector equations (1.1) in the generating approximation is integrable after (1.3) has been obtained. We assume that equation makes possible the effective construction of a quasi-periodic set of "purely induced" solutions of the form

$$u = u(\psi_1, \ldots, \psi_l, I_1, \ldots, I_l, \varphi_1, \ldots, \varphi_m, K_1, \ldots, K_m, \theta_1, \ldots, \theta_n, h_1, \ldots, h_n, \psi) \tag{1.9}$$

which is stable in-the-large.

The latter is ensured, if for instance that equation is linear and steady with respect to all components of u when $\mu = 0$.

Below, somewhat extending the problem, we use as the input system

$$x^{\cdot} = X(x, \psi, \mu) \tag{1.10}$$

which has all of the described properties and, consequently, admits for $\mu = 0$ the stable in-the-large set of quasi-periodic solutions

$$x = x(\psi, \psi_1, \ldots, \psi_l, I_1, \ldots, I_l, \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m, h_1, \ldots, h_n) \tag{1.11}$$

where, for simplicity, all constants that do not belong to anisochronous and quasi-periodic coordinates and denoted by $h_1, \ldots, h_n$.

The generating T-periodic solution has the property that anisochronous partial frequencies coincide with the perturbation frequency and the quasi-static actions vanish

$$K_s = 0, \ \omega_i(I_1, \ldots, I_l, 0, \ldots, 0) = \nu \tag{1.12}$$

Owing to the nondegeneracy of the steepness matrix (1.5), the relations (1.12) uniquely determine the anisochronous actions. Such T-periodic subset from (1.11) depends on constants $\alpha_1, \ldots, \alpha_l, \ h_1, \ldots, h_n$.

2. The criterion of existence and stability. The system of equations in variations of system (1.10)

$$y^{\cdot} = y\partial X/\partial x \tag{2.1}$$

close to the unknown T-periodic solution admits for $\mu = 0$, $l + n$ mutually independent solutions

$$(\partial x/\partial \alpha_i), \ i = 1, \ldots, l; \ (\partial x/\partial h_\gamma), \ \gamma = 1, \ldots, n \tag{2.2}$$

that are of the same order /1/, and l increasing solutions of the form

$$(\partial x/\partial \alpha_i)\, t + \vartheta_i \tag{2.3}$$

$$\vartheta_i = (e_{ij})^{-1}(\partial x/\partial I_j) \tag{2.4}$$

where $(e_{ij})^{-1}$ is a matrix inverse of (1.5), and the parentheses indicate here and in what follows that the respective quantity is calculated using the T-periodic generating solution.

Moreover, summation over recurring indices i, j, k from 1 to l, over index r from 1 to m, and over index δ from 1 to n is implied in (2.4).

System (2.1) also admits when $\mu = 0$ m pairs of complex conjugate mutually independent quasi-periodic partial solutions of the form

$$[(\partial x/\partial\lambda_s) \pm \sqrt{-1}\,(\partial x/\partial\mu_s)] \exp [\mp\sqrt{-1}\,(\varphi_s)] \tag{2.5}$$

If χ quasi-static frequencies $\Omega_1, \ldots, \Omega_\chi$ with $K_1 = \ldots = K_m$ become equal to a given value (Ω), then the respective χ partial quasi-periodic solutions of type (2.5) have the same frequency spectrum. Nevertheless, when $K_s = 0$, the respective partial solutions are mutually independent by virtue of the mutual independence of these frequencies. Thus when $\mu = 0$, the characteristic equation of system (2.1) has an n-multiple zero root with simple elementary divisors, $(2l)$-multiple zero root with quadratic simple divisors, and m pairs of pure imaginary indices of the form $\pm\sqrt{-1}\,(\Omega_s)$. Appearance of the latter is associated with the presence of quasi-static phases, and predetermines the essential difference of the considered degenerate problem from that of synchronization on all phases /1/.

Note also that all remaining characteristic indices of the solution, by virtue of stability of set (1.11), have negative real parts that are not small.

We introduce in the analysis the column vector z that satisfies the system

$$z^{\cdot} = -z\,(\partial X/\partial x) \tag{2.6}$$

conjugate of (2.1) when $\mu = 0$. We impose on the mutually independent T-periodic solutions $v_1, \ldots, v_n$ and $w_1, \ldots, w_l$ of (2.6) the following normalizing conditions

$$v_\delta\,(\partial x/\partial h_\gamma) = \delta_{\gamma\delta};\ w_j\vartheta_i = \delta_{ij} \tag{2.7}$$

The independent quasi-periodic partial solutions of (2.6)

$$(\rho_s \mp \sqrt{-1}\,\sigma_s) \exp [\pm\sqrt{-1}\,(\varphi_s)] \tag{2.8}$$

satisfy the following normalizing relations:

$$\begin{aligned} &\rho_r\,(\partial x/\partial\lambda_s) + \sigma_r\,(\partial x/\partial\mu_s) = \delta_{sr} \\ &\sigma_r\,(\partial x/\partial\lambda_s) - \rho_r\,(\partial x/\partial\mu_s) = 0 \end{aligned} \tag{2.9}$$

For the existence of T-periodic solution of (1.10) of the considered here type, which is analytic in μ in proximity of point $\mu = 0$ it is sufficient that the system of $l + n$ transcendental equations

$$\begin{aligned} P_\delta(\alpha_1, \ldots, \alpha_l, h_1, \ldots, h_n) &\equiv \int_0^T v_\delta\,(\partial X/\partial\mu)\,dt = 0 \\ R_j(\alpha_1, \ldots, \alpha_l, h_1, \ldots, h_n) &\equiv \int_0^T w_j\,(\partial X/\partial\mu)\,dt = 0 \end{aligned} \tag{2.10}$$

admits simple real solutions /2/.

Investigation of conditions of asymptotic stability in-the-small is based on the derivation of partial solutions of the form

$$y = \eta\,(t, \mu) \exp [\lambda\,(\mu)\,t] \tag{2.11}$$

of the system of equations in variations (2.1), where λ is the characteristic index and η is a column vector with T-periodic components. The derivation of solutions (2.11) which for $\mu = 0$ are transformed into (2.2) is similar to that carried out in /1/. It is shown in exactly the same way that there are n-indices $(\lambda = \lambda_1\mu + \lambda_2\mu^2 + \ldots)$ analytic in μ for which the first approximation λ_1 is determined from the conditions of nontriviality of solutions of the inhomogeneous system

$$\begin{aligned} a_j\partial P_\gamma/\partial\alpha_j + b_\delta\partial P_\gamma/\partial h_\delta &= \lambda_1 T b_\gamma \\ a_j\partial R_i/\partial\alpha_j + b_\delta\partial R_i/\partial h_\delta &= 0 \end{aligned} \tag{2.12}$$

There exist, moreover, l pairs of indices $(\lambda = \pm\mu^{1/2}\lambda_1 + \mu\lambda_2 + \mu^{3/2}\ldots)$ analytic in $\sqrt{\mu}$ for which the relations

$$\begin{aligned} &a_i\partial R_i/\partial\alpha_j = \lambda_1^2 a_i T \\ &\lambda_2 = \left[\frac{1}{2T}\,\partial R_j/\partial I_k\,(e_{ik})^{-1} + \frac{1}{\lambda_1^2 T}\,\partial R_j/\partial h_\delta\partial P_\delta/\partial\alpha_i - p_{ji}\right] a_i a_j^* \end{aligned} \tag{2.13}$$

$$p_{ji} = \int_0^T (w_j\, \partial x_1/\partial \alpha_i - \frac{t}{T}\, \partial R_j/\partial \alpha_i)\, dt \tag{2.14}$$

provide means for determining the first two approximations. The quantity x_1 is the T-periodic solution of the first approximation equation

$$x_1^{\cdot} = (\partial X/\partial x)\, x_1 + (\partial X/\partial \mu) \tag{2.15}$$

and a_i^* is the solution of the system conjugate of (2.13)

$$a_j^* \partial R_j/\partial \alpha_i = \lambda_1^2 T a_i^* \tag{2.16}$$

which satisfies the normalization condition $a_i a_i^* = 1$.

Essentially new stability criteria related to the presence of quasi-static phases exist and correspond to partial solutions (2.11) of system (2.1), which for $\mu = 0$ transform into (2.8). If for $K_1 = \ldots = K_m = 0$ χ quasi-static frequencies are equal (Ω), there exist quasi-periodic solutions that are analytic in μ and defined by the expansions

$$\begin{aligned} \eta &= a_\xi\, [(\partial x/\partial \lambda_\xi) - \sqrt{-1}\, (\partial x/\partial \mu_\xi)] + \mu\eta_1 + \mu^2 \ldots \\ \lambda &= \sqrt{-1}\, (\Omega) + \mu\lambda_1 + \mu^2 \ldots \end{aligned} \tag{2.17}$$

where a_ξ $(\xi = 1, \ldots, \chi)$ are some scalar constants, the vector functions $(\partial x/\partial \lambda_\xi)$, $(\partial x/\partial \mu_\xi)$ correspond to the quasi-static phase of the considered here "multiple" group with number ξ, and the successive approximations $\eta_1, \eta_2, \ldots$ are T-periodic in t. Here and subsequently summation from 1 to χ is implied by the recurrent indices ξ and ζ.

The substitution of series (2.17) into (2.2) yields the following first approximation equation:

$$\eta_1^{\cdot} = (\partial X/\partial x)\, \eta_1 + a_\xi\, [(\partial'/\partial\mu\, \partial X/\partial x) - \lambda_1 E]\, [(\partial x/\partial \lambda_\xi) - \sqrt{-1}\, (\partial x/\partial \mu_\xi)] - \sqrt{-1}\, (\Omega)\, \eta_1 \tag{2.18}$$

where the prime denotes total partial differentiation with respect to μ.

Expression for the first correction to the characteristic index λ_1 can be obtained from the condition of existence of a T-periodic solution of system (2.18) in the process of investigation of the homogeneous system

$$\begin{aligned} &a_\xi\, (q_{\xi\zeta} - \delta_{\xi\zeta}\lambda_1 T) = 0 \\ &q_{\xi\zeta} = \int_0^T [\rho_\zeta + \sqrt{-1}\, \sigma_\zeta]\, (\partial'/\partial\mu\, \partial X/\partial x)\, [(\partial x/\partial \lambda_\xi) - \sqrt{-1}\, (\partial x/\partial \mu_\xi)]\, dt \end{aligned} \tag{2.19}$$

For the asymptotic stability in-the-small of this T-periodic solutions it is, thus, sufficient that the following three groups of conditions are satisfied.

1°. The anisochronous stability conditions which decompose into two subgroups and specify that the corrections λ_1 and λ_2 determined in conformity with (2.13) must satisfy conditions that $\lambda_1^2 < 0$, $\lambda_2 < 0$.

2°. The isochronous stability conditions which imply the negativeness of real roots of the determinant of system (2.12).

3°. The quasi-static stability conditions which in conformity with (2.19) are charaterized by the fulfillment of inequalities $\operatorname{Re} \lambda_1 < 0$.

3. The case of the almost conservative system. The use of the obtained conditions of existence and stability in the most general form is somewhat complicated by the presence in them of periodic solutions of the conjugate system (2.6) which satisfy the completely defined normalization relations. For the almost conservative system (1.1) the system of equations in variations (2.1) is self-conjugate when $\mu = 0$. Owing to this, there exists correspondence between solutions of the system, which enables us to express the resulting conditions in terms of explicit functionals calculated with the use of the generating T-periodic solution.

With this in view we pass in the almost conservative input system (1.1) from the variables (p, q) to the new canonical variables $(\psi_i, I_i, \lambda_s, \mu_s, \theta_\gamma, h_\gamma)$ in conformity with (1.3)–(1.8). As the result we obtain

$$\begin{aligned} &I_i^{\cdot} = \mu Q \partial q/\partial \psi_i,\ \psi_i^{\cdot} = \omega_i - \mu Q \partial q/\partial I_i, (i = 1, \ldots, l) \\ &\mu_s^{\cdot} = \Omega_s \lambda_s - \mu Q \partial q/\partial \lambda_s,\ \lambda_s^{\cdot} = -\Omega_s \mu_s + \mu Q \partial q/\partial \mu_s,\ (s = 1, \ldots, m) \\ &h_\gamma^{\cdot} = \mu Q \partial q/\partial \theta_\gamma,\ \theta_\gamma^{\cdot} = -\mu Q \partial q/\partial h_\gamma\ (\gamma = 1, \ldots, n),\ u^{\cdot} = U \end{aligned}$$

The stable in-the-large set of quasi-periodic solutions of the generating system is

$$I_i = \text{const},\ \psi_i = \omega_i t + \alpha_i,\ \lambda_s = \sqrt{K_s}\cos\varphi_s,\quad \mu_s = \sqrt{K_s}\sin\varphi_s,\ K_s = \text{const},\ \varphi_s = \Omega_s t + \beta_s$$
$$h_\gamma = \text{const},\ \theta_\gamma = \text{const},\ u = u(I_i, \psi_i, \lambda_s, \mu_s, h_\gamma, \theta_\gamma)$$

By virtue of this we have $l + 2n$ periodic solutions of the system of equations in variations when $\mu = 0$

$$\delta I_i = 0,\ \delta\psi_i = \delta_{ij},\ \delta\lambda_s = 0,\ \delta\mu_s = 0,\ \delta h_\gamma = 0,\quad \delta\theta_\gamma = 0,\ \delta u = (\partial u/\partial\alpha_j)\ (j = 1, \ldots, l)$$
$$\delta I_i = 0,\ \delta\psi_i = 0,\ \delta\lambda_s = 0,\ \delta\mu_s = 0,\ \delta h_\gamma = \delta_{\gamma\delta},\quad \delta\theta_\gamma = 0,\ \delta u = (\partial u/\partial h_\delta)\ (\delta = 1, \ldots, n)$$
$$\delta I_i = 0,\ \delta\psi_i = 0,\ \delta\lambda_s = 0,\ \delta\mu_s = 0,\ \delta h_\gamma = 0,\quad \delta\theta_\gamma = \delta_{\gamma\delta},\ \delta u = (\partial u/\partial\theta_\delta)\ (\delta = 1, \ldots, n)$$

We denote the components of vector ϑ_j (see (2.4)) in this conservative problem by a prime

$$I_i' = (e_{ij})^{-1},\quad \psi_i' = 0,\quad \mu_s' = \lambda_s' = 0,\quad h_\gamma' = \theta_\gamma' = 0,\qquad u' = (e_{jk})^{-1}(\partial u/\partial I_k)$$

We denote variables on conjugate system (2.6) by an asterisk. As in the canonical case, that system becomes a system of equations in variations, after the inversion

$$\delta I_i \to \psi_i^*,\quad \delta\psi_i \to -I_i^*,\quad \delta\lambda_s \to \lambda_s^*,\quad \delta\mu_s \to \mu_s^*,\quad \delta h_\gamma \to h_\gamma^*,\quad \delta\theta_\gamma \to \theta_\gamma^*$$

and addition to the right-hand sides of these equations terms that are linear homogeneous forms of components of vector u^*.

As regards the last vector equation, it is of the form $u^{*\cdot} = -u^*(\partial U/\partial u)$.

It follows from the above that the periodic solutions of the conjugate system normalized in conformity with conditions (2.7) are of the form

$$I_i^* = (e_{ij}),\quad \psi_i^* = 0,\quad \lambda_r^* = \mu_r^* = 0,\quad h_\gamma^* = \theta_\gamma^* = 0,\quad u^* = 0,\qquad (j = 1, \ldots, l)$$
$$I_i^* = \psi_i^* = 0,\quad \lambda_r^* = \mu_r^* = 0,\quad h_\gamma^* = \delta_{\gamma\delta},\quad \theta_\gamma^* = 0,\quad u^* = 0,\qquad (\delta = 1, \ldots, n)$$
$$I_i^* = \psi_i^* = 0,\quad \lambda_r^* = \mu_r^* = 0,\quad h_\gamma^* = 0,\quad \theta_\gamma^* = \delta_{\gamma\delta},\quad u^* = 0,\qquad (\delta = 1, \ldots, n)$$

Equations (2.10) for the determination of parameters of the generating solution are

$$A_\gamma \equiv \int_0^T (Q\partial q/\partial\theta_\gamma)\,dt = 0,\quad B_\gamma \equiv -\int_0^T (Q\partial q/\partial h_\gamma)\,dt = 0,\quad (\gamma = 1, \ldots, n) \tag{3.1}$$
$$R_i \equiv (e_{ij})\int_0^T (Q\partial q/\partial\alpha_j)\,dt = 0\quad (i = 1, \ldots, l)$$

By virtue of the nondegeneracy of matrix (e_{ij}), the last l equations (3.1) are equivalent to the following simpler ones:

$$C_i \equiv \int_0^T (Q\partial q/\partial\alpha_i)\,dt = 0$$

The system of equations for the determination isochronous stability criteria (2.12) in the considered here almost conservative case reduces to the form

$$\begin{aligned} a_i\partial A_\delta/\partial\alpha_i + \partial A_\delta/\partial h_\gamma b_\gamma + c_\gamma\partial A_\delta/\partial\theta_\gamma &= \lambda_1 T c_\delta \\ a_i\partial B_\delta/\partial\alpha_i + b_\gamma\partial B_\delta/\partial h_\gamma + c_\gamma\partial B_\delta/\partial\theta_\gamma &= \lambda_1 T b_\delta \\ a_i\partial C_j/\partial\alpha_i + b_\gamma\partial C_j/\partial h_\gamma + c_\gamma\partial C_j/\partial\theta_\gamma &= 0 \end{aligned} \tag{3.2}$$

The anisochronous stability criteria for the first subgroup are determined in conformity with (2.13) by the system

$$(e_{jk})\, a_i\partial C_k/\partial\alpha_i = \lambda_1^2 T a_j \tag{3.3}$$

The system conjugate of (3.3) can be reduced to the form

$$(e_{ij})\, b_k\partial C_k/\partial\alpha_i = \lambda_1^2 T b_j$$

in which appears the new variable that satisfies the "weighted" normalization condition

$$a_i\,(e_{ij})^{-1}b_j = 1,\ a_i^* = (e_{ij})^{-1}b_j \tag{3.4}$$

The determination of anisochronous criteria of the second subgroup requires the expansion of the expression for matrix p_{ji} (2.14). Omitting intermediate calculations, we present the final formula

$$p_{ji} = \partial D_j/\partial\alpha_i,\quad D_j \equiv \int_0^T (Q\partial q/\partial I_j)\,dt \tag{3.5}$$

By virtue of (3.4) and (3.5) we thus have

$$\lambda_2 = \frac{1}{2T}\left[(\partial C_j/\partial I_k - \partial D_k/\partial \alpha_j)(l_{ki})^{-1} + \frac{1}{\lambda_1{}^2 T}(\partial C_j/\partial h_\gamma \partial A_\gamma/\partial \alpha_i + \partial C_j/\partial \theta_\gamma \partial B_\gamma/\partial \alpha_i)\right] a_i b_j$$

The components of vectors ρ_s and σ_s appearing in quasi-periodic solutions of the conjugate system of type (2.8) that satisfy normalization conditions of type (2.9) are denoted below by parantheses with asterisks, respectively, above and below

$$(I_i)^* = (\psi_i)^* = 0, \quad (\lambda_s)^* = \frac{1}{2}\delta_{sr}, \quad (\mu_s)^* = 0, \quad (h_\gamma)^* = (\theta_\gamma)^* = 0, \quad (u)^* = 0$$

$$(I_i)_* = (\psi_i)_* = 0, \quad (\lambda_s)_* = 0, \quad (\mu_s)_* = \frac{1}{2}\delta_{sr}, \quad (h_\gamma)_* = (\theta_\gamma)_* = 0, \quad (u)_* = 0$$

In consequence of the above Eqs.(2.19) that define the quasi-static stability criteria which correspond to the χ-multiple generating value of (Ω) assume the form

$$a_\xi(q_{\xi\zeta} - \delta_{\xi\zeta}\lambda_1 T) = 0$$

$$q_{\xi\zeta} = \int_0^T [(\partial/\partial\lambda_\xi Q \partial q/\partial\mu_\zeta - \partial/\partial\mu_\xi Q \partial q/\partial\lambda_\zeta) - \sqrt{-1}\,(\partial/\partial\mu_\xi Q \partial q/\partial\mu_\zeta + \partial/\partial\lambda_\xi Q \partial q/\partial\lambda_\zeta)]\, dt$$

Fulfillment of the quasi-static stability conditions in the linear problem of weak nonresonant interaction of identical linear oscillators

$$q_s^{\cdot\cdot} + \Omega^2 q_s = \mu\,(b_{sr} q_r^{\cdot} + a_{sr} q_r + h_s \sin \omega t) \tag{3.6}$$

where the numbers Ω and ω are mutually incommensurable and the matrix components b_{sr} and a_{sr} are constant, reduces to the fulfillment of inequalities $\lambda_1 < 0$ in which λ_I is determined in the course of investigation of the homogeneous system of linear equations

$$\left[\frac{1}{2}\left(b_{sr} - \sqrt{-1}\,\frac{d_{sr}}{\Omega}\right) - \lambda_1 \delta_{sr}\right] a_r = 0$$

Thus in the considered here nonresonance case only quasi-static criteria are important, since they are intrinsically independent of phase difference $\Omega - \omega = O(1)$, and ensure stability of the zero solution of the homogeneous part of (3.6).

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